NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1252

SOME STUDIES ON THE FLOW OF A GAS IN THE REGION OF TRANSITION THROUGH THE VELOCITY OF SOUND

By I. A. Kiebel

TRANSLATION

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SOME STUDIES ON THE FLOW OF A GAS IN THE REGION OF TRANSITION THROUGH THE VELOCITY OF SOUND*

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The two-dimensional motion of an incompressible fluid about a closed contour with a definite velocity in magnitude and direction at infinity is considered. If, without changing the direction of the velocity at infinity, the magnitude is increased, the configuration of the streamlines remains unchanged and only the numbering of the stream function changes. There exists only one family of curves that can serve as streamlines in the incompressible flow about a given contour (at a given angle of attack); for example, the contour of an airplane wing. The case is quite different with a compressible fluid. For the components v_x and v_y of the velocity along the x- and y-axes for incompressible flow

$$\frac{v_x}{v_\infty} = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}$$

$$\frac{v_y}{v_\infty} = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

(v_∞ is the magnitude of the velocity at infinity, φ the velocity potential, and ψ the stream function) so that for ψ there is obtained a simple Laplace equation. For a compressible gas moving adiabatically, there must be obtained

$$\frac{v_x}{v_\infty} = \frac{\partial\varphi}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y}$$

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*"Nekotorye Raboty o Techeniakh Gaza v Oblasti Perekhoda Cherez Skorost Zvuka." Izvestia Akademii Nauk SSSR, No. 3, 1947, pp. 253-259.

$$\frac{v_y}{v_\infty} = \frac{\partial\varphi}{\partial y} = -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x}$$

where

$$\frac{\rho}{\rho_0} = \left[1 - \frac{k-1}{k+1}\frac{v^2}{v_\infty^2}\left(\frac{v_\infty}{a_*}\right)^2\right]^{\frac{1}{k-1}}$$

v is the magnitude of the velocity, a_* the critical velocity, and k the ratio of specific heats. The equation obtained for the stream function will now contain as a parameter $(v_\infty/a_*)^2$, so that to one profile there will correspond an infinity of streamlines representing this profile for different values of the velocity at infinity.

The change in shape of the streamlines is particularly sharp when in the flow plane (still having a subsonic velocity at infinity) supersonic zones arise. A new specific difficulty now arises; namely, that the flow of a compressible fluid has two singularities as compared with the motion of an incompressible fluid. In the first place in the compressible fluid in contrast with the incompressible fluid, infinitely large velocities are impossible (the maximum possible velocity is $a_* \sqrt{(k+1)/(k-1)}$); in the second place in the supersonic flow of a gas, the stream tubes, in contrast to what holds for the incompressible fluid, expand with increasing velocity. This last circumstance leads to the fact that in the supersonic zone the streamlines will diverge and move away from the boundary, whereas in the subsonic zone the streamlines will converge on approaching the supersonic zone. It may be expected that for a given contour there will exist velocities at infinity for which both these laws cannot be satisfied. The finiteness of the velocity on the other hand leads to the fact that where the solution for the incompressible flow gives infinity velocities (for example, in the flow about an edge), the solution for the compressible flow either does not exist or the existing strealines do not form a sharp angle. Mathematically expressed, in the supersonic zone points and entire lines may appear on which the derivatives of the velocities become infinite. Such solutions of the equations of gas dynamics, although existing formally, have no physical meaning and cannot be realized. In these cases, the motion apparently so readjusts itself that a line of strong discontinuity arises and the solution from the very beginning must be sought not in the form of continuous irrotational flow but in the form of a motion in which there is a surface of discontinuity, the shape and position of which are unknown initially and after passing through which the motion becomes rotational. Tests show that such jumps are, as a rule, actually found on the wing in the supersonic zone (the wing moves with subsonic velocity).

A complete theory of the transition through the velocity of sound has not, as yet, been developed. The problem of the transition through the velocity of sound is one of the primary problems of aerodynamics.

To N. E. Joukowsky belongs the credit for the known modification of the method of Kirchhoff, where in the study of the motion of an incompressible fluid there are used in place of the coordinates x and y the magnitudes θ (the angle of inclination of the velocity) and $Z = \ln\ v_\infty/v$ as the independent variables.

Expressed in these variables, the equations for the stream function ψ and the velocity potential φ (the motion is considered irrotational) the following form is assumed:

$$\frac{\partial\varphi}{\partial\theta} = -\frac{\partial\psi}{\partial Z}$$

$$\frac{\partial\varphi}{\partial Z} = \frac{\partial\psi}{\partial\theta}$$

so that for the determination of ψ, the Laplace equation is obtained

$$\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial Z^2} = 0$$

In the case of a compressible fluid, a natural generalization of the variable Z is the function S of the velocity, employed by S. A. Christianovich in his fundamental work (reference 1) in the form

$$S = \int \frac{\sqrt{1 - M^2}}{v}\, dv + \text{constant}$$

where

$$M = \frac{v}{a}$$

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a velocity of sound

The equations of S. A. Christianovich for the compressible fluid may be written in the form

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial\theta} &= -\sqrt{K}\,\frac{\partial\psi}{\partial\bar{S}} \\ \frac{\partial\varphi}{\partial\bar{S}} &= \sqrt{K}\,\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{1}$$

where θ, as before, is the angle of inclination of the velocity,

$$\sqrt{K} = \frac{\rho_0}{\rho}\sqrt{1 - M^2}$$

and

$$\bar{S} = \int_v^{a_*} \sqrt{1 - M^2}\,\frac{dv}{v}$$

For the supersonic region, it is convenient to introduce, as was done in another report of S. A. Christianovich (reference 2), the functions χ and $\bar{\sigma}$ from the relations

$$\sqrt{\chi} = \frac{\rho_0}{\rho}\sqrt{M^2 - 1}$$

$$\bar{\bar{\sigma}} = \int_{a_*}^{v} \sqrt{M^2 - 1}\,\frac{dv}{dv}$$ [1].

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1 $\bar{\bar{\sigma}} = 1 - \bar{\sigma}$ from reference 2.

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and the equations for φ and ψ then assume the form

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= \sqrt{\chi}\,\frac{\partial\psi}{\partial\overline{\sigma}} \\ \frac{\partial\varphi}{\partial\overline{\sigma}} &= \sqrt{\chi}\,\frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \qquad (2)$$

The functions K, $\overline{S}$, χ, and $\overline{\sigma}$ depend, in accordance with the Bernoulli law, only on the velocity, and

$$\sqrt{K} = t\left(\frac{h^2 - t^2}{h^2 - 1}\right)^{\frac{h^2 - 1}{2}}$$

$$\overline{S} = \frac{1}{2}\ln\left[\frac{1 + t}{1 - t}\left(\frac{h - t}{h + t}\right)^h\right]$$

where

$$h^2 = \frac{k + 1}{k - 1}$$

$$t = \sqrt{1 - M^2}$$

and

$$\sqrt{\chi} = \overline{t}\left(\frac{h^2 + \overline{t}^2}{h^2 - 1}\right)^{\frac{h^2 - 1}{2}}$$

$$\overline{\sigma} = h \operatorname{arc\,tan}\frac{\overline{t}}{h} - \operatorname{arc\,tan}\overline{t}$$

where

$$\overline{t} = \sqrt{M^2 - 1}$$

The function $\sqrt{K}$ through the intermediary of the parameter t depends on $\overline{S}$; the function $\sqrt{X}$ through the intermediary of $\overline{t}$ depends on $\overline{\sigma}$. In figure 1, the values of $\overline{S}$ are in the direction of the positive horizontal axis and the values of $\sqrt{K}$ along the positive vertical axis; on the negative horizontal axis are the values of $-\overline{\sigma}$ and along the negative vertical axis, $-\sqrt{X}$. As the velocities increase from 0 $(M = 0)$ to a_* $(M = 1)$, $\overline{S}$ decreases from ∞ to 0. The value of $\sqrt{K}$ up to very small values of $\overline{S}$ will be very near unity and only at $\overline{S} = 0$ does $\sqrt{K}$ rapidly tend to zero. Further, when $\overline{\sigma}$ increases from zero to its limiting value equal to $\frac{\pi}{2}(h-1)$ the function $\sqrt{X}$ will asymptotically tend to infinity (for $\overline{\sigma} = -\frac{3}{2k} + \sqrt{\frac{9}{4k^2} + \frac{k+1}{k}}$ the curve has a point of inflection).

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When φ is eliminated in the subsonic region, the following equation is eliminated

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\overline{S}^2} + \frac{d\ \ln\sqrt{K}}{d\overline{S}}\frac{\partial\psi}{\partial\overline{S}} = 0$$

and in the supersonic region

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\overline{\sigma}^2} + \frac{d\ \ln\sqrt{X}}{d\overline{\sigma}}\frac{\partial\psi}{\partial\overline{\sigma}} = 0$$

On the basis of the behavior of the function $\sqrt{K}$, in equations (1) $\sqrt{K} = \text{constant}$ may be set to very small values of $\overline{S}$. The equation of Laplace is then obtained, which is the first approximation of S. A. Christianovich (references 1 and 3). In one of his later papers, Christianovich showed (reference 4) that for the supersonic region, that is, where equations (2) must be taken, starting from small $\overline{\sigma}$ and further on, the function $\sqrt{X}$ may be approximated in the form of pieces of curves of the second order (first a parabola and then a hyperbola). Equations (2) then become the equation of Darboux and may be simply integrated to the end.

The analysis is, however, most difficult of all for the narrow region corresponding to the values of $\overline{S}$ and $\overline{\sigma}$ near zero. This region is shown to a magnified scale in figure 2. This is the region of transition through the sound velocity, which was previously discussed. It is first necessary to mention the investigation of F. I. Frankl (references 5, 6), who first for the region of transition separated out the principal term in the equations for φ and ψ and obtained for the principal term the equation of Tricomi.[2] Frankl gives a profound analysis of the phenomenon of the transition through the velocity of sound and in particular investigated the condition at which in the Laval nozzle a transition without the occurrence of a surface of discontinuity is possible. Recently his coworker, S. V. Falkovich (references 7 and 8), on the basis of Frankl's work notes the fact that the function $\sqrt{K}$ about $\overline{S} = 0$ can be devloped into a series of the form $\sqrt{K} = b_1 \sqrt[3]{\overline{S}} + \ldots$, where b_1 depends on $k\left(b_1 = \sqrt[3]{\frac{h^2}{3(h^2-1)}}\right)$; similarly $\sqrt{\chi} = b_1 \sqrt[3]{\overline{\sigma}} + \ldots$. Falkovich then assumes near the transition line the following equation for ψ:

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\overline{S}^2} + \frac{1}{3\overline{S}}\frac{\partial\psi}{\partial\overline{S}} = 0 \tag{3}$$

Introducing in place of θ and $\overline{S}$ plane bipolar coordinates α and β from the relations

$$\alpha = \ln\sqrt{\frac{\theta^2 + (\overline{S} + S_0)^2}{\theta^2 + (\overline{S} - S_0)^2}}$$

$$\beta = \arctan\frac{2S_0\theta}{\theta^2 + \overline{S}^2 - S_0{}^2}$$

where S_0 is the value of $\overline{S}$ corresponding to the velocity at the the point where $\theta = 0$ (the velocity at infinity), Falkovich was able to construct a number of important particular solutions of

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[2]Frankl introduced as independent variables θ and

$\eta = \left(\frac{3}{2}\overline{S}\right)^{\frac{1}{3}} = \left(\frac{3}{2}\overline{\sigma}\right)^{\frac{1}{3}}$.

equation (3). Owing to the work of Falkovich, it is possible to construct for the motion near the velocity of sound simple solutions having singularities of the type that arise in an incompressible fluid in the problem of the flow with circulation.

The occurrence of a surface of discontinuity in the supersonic region greatly complicates the solution of the problem. After passing through the surface of discontinuity, the motion becomes rotational - the velocity potential does not apply. For the stream function, the following complicated equation is obtained:

$$\left.\begin{aligned} v\frac{\partial}{\partial v}\left\{\frac{\rho_0}{\rho}\frac{1}{1+v\frac{N}{M^2}\frac{\partial}{\partial v}}\left[\frac{\partial\psi}{\partial v}-\frac{1-M^2}{vM^2}N\left(\frac{\partial\psi}{\partial\theta}\right)^2\right]\right\}+\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial v}+ \\ \frac{\rho_0}{\rho}\frac{1-M^2}{v}\frac{\partial^2\psi}{\partial\theta^2}=0 \end{aligned}\right\} \tag{4}$$

where

$$N=\frac{1}{k-1}\frac{d\ln\vartheta(\psi)}{d\psi}$$

where ϑ, a function only of ψ (constant in the irrotational case), is given by the value $\vartheta=\frac{1}{\rho}p^{\frac{1}{k}}$ and the form of this function depends on the shape of the surface of discontinuity; ρ_0/ρ and M have the previous values

$$\frac{\rho}{\rho_0}=\left(1-\frac{v^2}{h^2a_*^2}\right)^{\frac{1}{k-1}}$$

3.

$$M^2=\frac{2}{k+1}\frac{\frac{v^2}{a_*^2}}{1-\frac{1}{h^2}\frac{v^2}{a_*^2}}$$

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[3]As in the irrotational case, a_* remains constant.

In studying the rotational motions, it would apparently be more convenient to proceed differently; namely, to take as independent variables one of the coordinates, for example, x and the stream function ψ. By making use of these independent variables and introducing as the required function $\int p\,dx$ (the integration is taken for fixed ψ), a number of particular exact solutions of the rotational problem (reference 9) are obtained. In particular, a solution is constructed with a transition through the velocity of sound. At the AN SSSR Institute of Mechanics, investigations are being continued on the transition zone in both the rotational and irrotational cases.

Translated by S. Reiss,
National Advisory Committee
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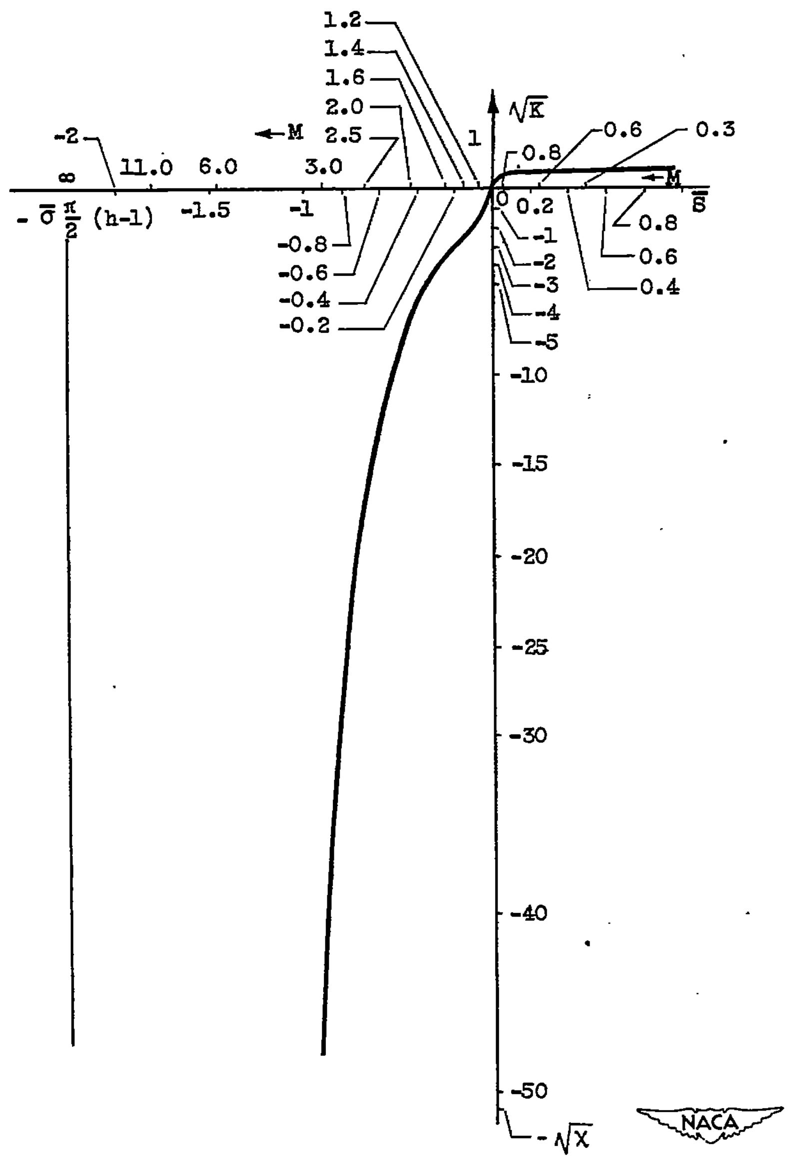
1.2
1.4
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-2
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Figure 1.

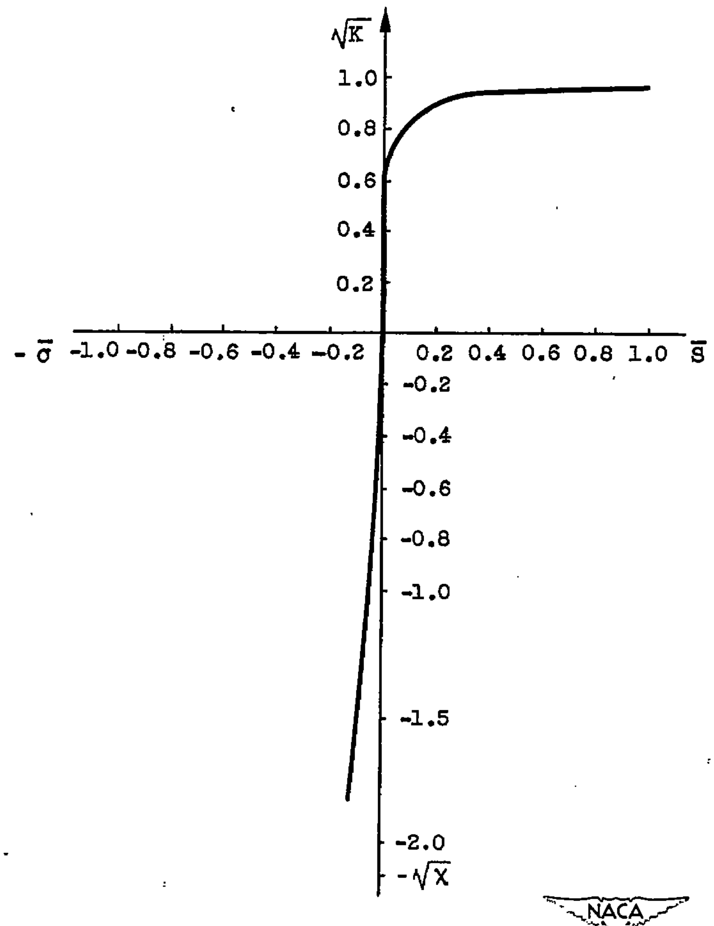

Figure 2.

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